# Rigidification

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**Def** $H$ flat fin pres. separated gp scheme $/S$

$\mathcal{M}$ alg stack $/S$

$\mathcal{M}$ has an H-2-structure if $\forall \xi \in \mathcal{M}(T)$,

$\exists$ embedding $\imath_\xi : H(T) \hookrightarrow \mathrm{Aut}(\xi)$

sth $\forall \phi : \xi \to \eta$ in $\mathcal{M}$ over $f : Z \to T$

$$\begin{tikzcd} H(T) \arrow[r] \arrow[d, hook, "\imath_\eta"] & H(Z) \arrow[d, hook, "\imath_\xi"] \\ \mathrm{Aut}(\eta) \arrow[r] & \mathrm{Aut}(\xi) \end{tikzcd}$$

**Rem.** if $\xi = \eta$, $\phi \in \mathrm{Aut}_T(\xi)$,

$$\begin{tikzcd} H(T) \arrow[r, "id"] \arrow[d, hook] & H(T) \arrow[d, hook] \\ \mathrm{Aut}_T(\xi) \arrow[r, "\phi(\cdot)\phi^{-1}"] & \mathrm{Aut}_T(\xi) \end{tikzcd}$$

$\Longrightarrow$ $H(T)$ in center of $\mathrm{Aut}_T(\xi)$

**Example** $\mathcal{M} \to T$ an $H$-gerbe (def. like for $\mathbb{G}_m$-gerbe)

$\Longrightarrow$ $\mathcal{M}$ has an H-2-structure,

with $H(T) = \mathrm{Aut}_T(\xi)$ $\forall \xi \in \mathcal{M}(T)$.

**Thm** (Abramovich, Corti, Vistoli) $\mathcal{M}$ alg stack with H-2-structure

$\Longrightarrow$ $\exists$ smooth surjective fin. presented morphism of alg stacks

$\mathcal{M} \longrightarrow \mathcal{M} /\!\!/ H$ sth:

(1) $\forall \xi \in \mathcal{M}(T)$ with image $\eta \in (\mathcal{M} /\!\!/ H)(T)$,

$H(T) \subset \ker(\mathrm{Aut}_T(\xi) \to \mathrm{Aut}_T(\eta))$.

(2) $\mathcal{M} \to \mathcal{M} /\!\!/ H$ is universal with this property (1).

(3) if $T = \mathrm{Spec}\, \bar{k}$, $\bar{k}$ alg closed field, then $\mathrm{Aut}_T(\eta) = \mathrm{Aut}_T(\xi)/H$.

(4) A moduli space for $\mathcal{M}$ is also a moduli space for $\mathcal{M} /\!\!/ H$.

(5) Furthermore, if $\mathcal{M}$ is Deligne-Mumford stack, then $\mathcal{M} /\!\!/ H$ is DM as well & $\mathcal{M} \to \mathcal{M} /\!\!/_H$ is étale.

## Quasicoherent sheaves on gerbes

$\mathcal{M} \to \mathcal{N}$ $\mathbb{G}_m$-gerbe

A qcoh sheaf $\mathcal{F}$ on $\mathcal{M}$ is given by

(1) $\forall X \xrightarrow{p} \mathcal{M}$, a qcoh sheaf $\mathcal{F}_p$

(2) $\forall$
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \mathcal{M} & \end{tikzcd}$$
with $\varphi : q \circ f \simeq p$,

an iso $\vartheta_{f,\varphi} : f^* \mathcal{F}_p \xrightarrow{\sim} \mathcal{F}_q$.

In particular
$$\begin{tikzcd} X \arrow[rr, "id"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & \mathcal{M} & \end{tikzcd}$$

$\varphi \in Aut(p/\mathcal{N}) \simeq \mathbb{G}_m$

$\rightsquigarrow$ get $\vartheta_\varphi : \mathcal{F}_p \xrightarrow{\sim} \mathcal{F}_p$ in $Aut(\mathcal{F}_p)$

$\rightsquigarrow$ $\mathbb{G}_m$-action $\mathbb{G}_m \times \mathcal{F} \to \mathcal{F}$

$\rightsquigarrow$ decomposition $\mathcal{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^{(d)}$

where $\mathbb{G}_m$ acts on $\mathcal{F}^d$ by $(t \mapsto t^d)$

$\rightsquigarrow$ $D^b(QCoh(\mathcal{M})) \simeq D^b(QCoh(\mathcal{M}))^{(d)}$

Note: If $\mathcal{M} \to \mathcal{N}$ is a trivial gerbe, then $\exists$ 1-twisted line bdle $\mathcal{L}$ on $\mathcal{M}$ so $D^b(QCoh(\mathcal{M}))^{(d)} \simeq D^b(QCoh(\mathcal{M}))^{(0)}$ $\forall d$

(this usually fails for nontrivial gerbes!)

# Picard stacks

**Def.** A Picard stack is a stack $\mathcal{P}$ with a bifunctor $\otimes : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}$ with associativity & commutativity constraints

$$a : \otimes \circ (\otimes \times 1) \xrightarrow{\sim} \otimes \circ (1 \times \otimes)$$

$$c : \otimes \xrightarrow{\sim} \otimes \circ \text{flip}$$

with $c_{X,X} = id$

sth $\forall X$, $\mathcal{P}(X)$ is a Picard groupoid (ie symmetric monoidal groupoid where all elements are invertible)

Let $PS :=$ cat of Picard stacks.

**Ex.** $B\mathbb{G}_m$ is a Picard stack.

Homomorphisms: $\mathcal{P}_1, \mathcal{P}_2$ Picard stacks

$\rightsquigarrow$ $\mathrm{Hom}_{PS}(\mathcal{P}_1, \mathcal{P}_2)$ is a category, again a Picard stack (via $\otimes$ on $\mathcal{P}_2$)!

Short exact sequences: Let $a : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be a hom. of Picard stacks.

Put $\ker(a) := \mathcal{P}_1 \underset{\mathcal{P}_2}{\times} \{e\}$, again a Picard stack.

A left exact sequence of Picard stacks is a sequence

$$1 \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_2 \xrightarrow{a} \mathcal{P}_3 \quad \text{with } \mathcal{P}_1 \simeq \ker(a).$$

If in addition $a$ is essentially surjective, (locally in fppf) we say the sequence is exact.

# Duality of Picard stacks

**Def** $\mathcal{P}$ Picard stack

$$\mathcal{P}^\vee := \mathrm{Hom}_{PS}(\mathcal{P}, B\mathbb{G}_m)$$

Examples (1) $\mathcal{P} = \mathbb{Z} \rightsquigarrow \mathcal{P}^\vee = \mathrm{Hom}_{PS}(\mathbb{Z}, B\mathbb{G}_m) = B\mathbb{G}_m$,

more generally:

$\mathcal{P} = \Gamma$, for $\Gamma$ fin gen abelian gp

then $\underbrace{\mathcal{P}^\vee(X)}_{\text{connected}} = \mathrm{Hom}_{\text{groupoids}}(\Gamma, \langle \text{line bdles on } X \rangle)$

$$= \mathrm{Hom}(\Gamma, \langle (\textstyle\coprod_i X_i \to X, \underbrace{g_{ij} \in \mathbb{G}_m(X_i \times_X X_j)}_{\text{cocycle}}) \rangle)$$

$$= \langle (\textstyle\coprod_i X_i \to X, \varphi_{ij} \in \mathrm{Hom}(\Gamma, \mathbb{G}_m(X_i \times_X X_j))) \rangle$$

$$= B\Gamma^\vee \quad \text{for} \quad \Gamma^\vee := \mathrm{Hom}(\Gamma, \mathbb{G}_m).$$

(2) $\mathcal{P} = B\mathbb{G}_m \rightsquigarrow$ Claim: $\mathcal{P}^\vee = \mathbb{Z}$

Must show: $\forall \varphi \in \mathrm{Hom}(B\mathbb{G}_m, B\mathbb{G}_m)$, $\exists!\, n \in \mathbb{Z}$

sth $\varphi_X$ is canonically isomorphic to $\mathcal{L} \mapsto \mathcal{L}^{\otimes n}$.

$\varphi_X$ tensor functor $\Rightarrow$ $\forall$ line bdle $\mathcal{L}$,

get line bdle $\varphi_X(\mathcal{L})$,

$\forall$ iso of line bdles $\mathcal{L} \xrightarrow{\sim} \mathcal{L}'$

get $\varphi_X(\mathcal{L}) \xrightarrow{\sim} \varphi_X(\mathcal{L}')$

in particular, get a homomorphism $\mathrm{Aut}(\mathcal{L}) \to \mathrm{Aut}(\varphi_X(\mathcal{L}))$

↳ since tensor functor

In particular: $\varphi_X(\mathcal{O}_X) \cong \mathcal{O}_X$ (unit element maps to unit element for a tensor fctor)

$\rightsquigarrow$ induces homomorphism $\eta_X : \mathrm{Aut}(\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X^*) \to \Gamma(X, \mathcal{O}_X^*)$

compatible with pullbacks:

$$\begin{tikzcd} \Gamma(X', \mathcal{O}_{X'}^*) \arrow[r, "\eta_{X'}"] \arrow[d, "f^*"] & \Gamma(X', \mathcal{O}_{X'}^*) \arrow[d, "f^*"] \\ \Gamma(X, \mathcal{O}_X^*) \arrow[r, "\eta_X"] & \Gamma(X, \mathcal{O}_X^*) \end{tikzcd}$$

for $X \to X'$.

Every line bdle locally trivial $\rightsquigarrow$ $\varphi$ determined by $(\eta_x)_x$

$(\eta_x)_x$ defines a homomorphism of gp schemes $\mathbb{G}_m \to \mathbb{G}_m$,

hence is $\in \operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m) \xrightarrow{\sim} \mathbb{Z}$

$(t \mapsto t^n) \longleftrightarrow n$ $\Longrightarrow$ claim.

Analogously: $(BG)^\vee = G^\vee$ for $G$ a group of multiplicative type

$$G^\vee = \operatorname{Hom}(G, \mathbb{G}_m)$$

(3) $P = A$ abelian variety $\rightsquigarrow$ $A^\vee$ is the dual abelian variety since $\operatorname{Hom}(A, B\mathbb{G}_m)$ classifies multiplicative $\{$line bdles on $A$

$\hookrightarrow$ (i.e. $m^*\mathcal{L} = \mathcal{L} \boxtimes \mathcal{L}$, i.e. $\deg(\mathcal{L}) = 0$)

(4) $C/k$ smooth proj curve, geom. connected

Picard scheme $\underline{\operatorname{Pic}}_C$ represents functor which is fppf sheafification of $X \to \operatorname{Pic}(X \times C)$. For all $X$, have exact sequence

$$0 \to \operatorname{Pic}(X) \to \operatorname{Pic}(X \times C) \to \underline{\operatorname{Pic}}_C(X).$$

$\underline{\operatorname{Pic}}_C$ is coarse moduli space for the Picard stack $\operatorname{Pic}_C$

get: $\kappa : \operatorname{Pic}_C \to \underline{\operatorname{Pic}}_C$, a $\mathbb{G}_m$-gerbe.

Natural decomposition $\operatorname{Pic}_C = \bigoplus_{d \in \mathbb{Z}} \operatorname{Pic}_C^d$

$$\underline{\operatorname{Pic}}_C = \bigoplus_{d \in \mathbb{Z}} \underline{\operatorname{Pic}}_C^d$$

Now assume $C$ has a rational pt $s : \operatorname{Spec} k \to C$

$\rightsquigarrow \mathcal{O}(s)$ degree 1

get
$$\begin{matrix} \mathrm{Pic}^0_C \xrightarrow{\sim} \mathrm{Pic}^d_C \\ \underline{\mathrm{Pic}}^0_C \xrightarrow{\sim} \underline{\mathrm{Pic}}^d_C \end{matrix} \quad \rightsquigarrow \quad \mathrm{Pic}_C \cong \mathrm{Pic}^0_C \times \mathbb{Z}$$

$s$ induces
$$\ker(\mathrm{Pic}(X\times C) \xrightarrow{s^*} \mathrm{Pic}(X)) \xrightarrow{\sim} \underline{\mathrm{Pic}}_C(X)$$
$$0 \to \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X\times C) \longrightarrow \underline{\mathrm{Pic}}_C(X) \to 0$$

$\Longrightarrow$ section to $\mathrm{Pic}_C \to \underline{\mathrm{Pic}}_C$

$\Longrightarrow$ $\mathrm{Pic}_C$ is a trivial gerbe over $\underline{\mathrm{Pic}}_C$

$$\Longrightarrow \mathrm{Pic}_C \cong \underline{\mathrm{Pic}}_C \times B\mathbb{G}_m$$
$$\cong \underline{\mathrm{Pic}}^0_C \times \mathbb{Z} \times B\mathbb{G}_m,$$

$$\mathrm{Pic}_C^\vee \cong \underline{\mathrm{Pic}}^0_C \times B\mathbb{G}_m \times \mathbb{Z} \qquad \Longrightarrow \mathrm{Pic}_C \text{ selfdual.}$$

(idem if $s$ doesn't exist: use that $s$ exists locally & iso doesn't depend on $s$ so that it glues).

<u>Def</u> $\mathcal{P}$ Picard stack is called <u>dualizable</u> if $\mathcal{P} \longrightarrow (\mathcal{P}^\vee)^\vee$ is an iso.

<u>Fourier - Mukai trafo</u>

$$\begin{matrix} & \mathcal{P}\times\mathcal{P}^\vee & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathcal{P} & & \mathcal{P}^\vee \end{matrix}$$

If $\mathcal{P}$ is a dualizable Picard stack,
$\exists$ Poincaré line bdle $\mathcal{L}_\mathcal{P}$ on $\mathcal{P}\times\mathcal{P}^\vee$ (since $\mathcal{P}^\vee$ classifies line bdles on $\mathcal{P}$)

Fourier Mukai functor
$$\Phi_\mathcal{P} : D^b(QCoh(\mathcal{P})) \longrightarrow D^b(QCoh(\mathcal{P}^\vee))$$
$$\mathcal{F} \longmapsto Rp_{2*}(Lp_1^*\mathcal{F} \otimes \mathcal{L}_\mathcal{P}).$$

this is an equivalence of cat$^s$ for all the above examples!

## Beilinson's 1-motives

**Def** A Picard stack $\mathcal{P}$ is called a Beilinson 1-motive if $\exists$ filtration $0 = W_{-1} \subset W_0 \subset W_1 \subset W_2 = \mathcal{P}$

s.th. (i) $Gr_0^W \simeq BG$, $G$ of mult type

(ii) $Gr_1^W \simeq A$ abelian variety

(iii) $Gr_2^W \simeq \Gamma$ fin. gen. abelian gp.

**Lemma.** Locally any such $\mathcal{P}$ is $\simeq A \times BG \times \Gamma$ (but not globally).

**Thm** (Arinkin) $\mathcal{P}$ Beilinson 1-motive

$\Longrightarrow$ $\mathcal{P}$ dualizable

& $\Phi_{\mathcal{P}} : D^b(QCoh(\mathcal{P})) \xrightarrow{\sim} D^b(QCoh(\mathcal{P}^\vee))$ equiv. of cat$^s$.